Midesmestral exam
Algebra III
B.Math. Hons. IInd year
First semester 2013
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In what follows, A denotes a non-zero commutative ring with unity unless stated otherwise explicitly.

Q 1. (8 marks)
State and prove the Chinese remainder theorem for A.

OR

Show that the set of nilpotent elements of A equals the intersection of all the prime ideals of A.

Q 2. (8 marks)
If every prime ideal of A is finitely generated, prove that every ideal of A is finitely generated.

OR

Show that the set

$$Z := \{x \in A : xy = 0 \quad for \quad some \quad 0 \neq y \in A\}$$

contains a prime ideal of A.
Hint: Look at the complement of Z.

Q 3. (8 marks)
Prove that an integral domain in which each non-zero, non-unit is a finite product of prime elements, must be a UFD.

OR

If p is a prime of the form $3k + 1$, show that p can be expressed as $a^2 + 3b^2$ for some integers a, b.
Hint: Use the fact that $\mathbf{Z}[\omega]$ is a UFD.

Q 4. (8 marks)
Let $\overline{\mathbf{Z}}$ denote the ring of all algebraic integers (complex numbers which are roots of a monic integer polynomial). Prove that there exists a strictly ascending chain of ideals.
Hint: Look at roots of unity.

OR

Show that

$$\overline{\mathbf{Z}} \cap \mathbf{Q}(\sqrt{2013}) = \mathbf{Z}\left[\frac{1+\sqrt{2013}}{2}\right]$$

where $\overline{\mathbf{Z}}$ is as above and $\mathbf{Q}(\sqrt{2013}) = \{\alpha + \beta\sqrt{2013} : \alpha, \beta \in \mathbf{Q}\}$.

Q 5. (8 marks)
If $f, g \in \mathbf{C}[X, Y]$ are so that

$$\{(z_1, z_2) \in \mathbf{C}^2 : f(z_1, z_2) = 0 = g(z_1, z_2)\} = \emptyset$$

prove that there exist $h_, h_2 \in \mathbf{C}[X, Y]$ such that $1 = fh_1 + gh_2$.

OR

(i) Give an example of a non-zero ideal of $C[0, 1]$ which is not a prime ideal.
(ii) Give an example of a prime ideal of $C[0, 1]$ which is not maximal.
Hint for (ii): Consider the multiplicative set S of all functions in $C[0, 1]$ which are represented by monic polynomials.

Q 6. (20 marks)
(i) Prove that $X^4 + 3X^2 + 7X + 4$ is irreducible over $\mathbf{Z}$.
(ii) If $R \subseteq S \subseteq K$ where R is a PID, K its quotient field and S a subring of K containing 1, prove that S must be a PID.
(iii) Let n be a positive integer. Suppose $f = \sum_{i=0}^{r} a_i X^i, g = \sum_{j=0}^{s} b_j X^j \in \mathbf{Z}/n\mathbf{Z}[X]$ are such that $fg = 0$. Show that $a_i b_j = 0$ for all i, j.
(iv) Let A be finite. Then, show that each prime ideal must be maximal.
(v) If I, J are co-maximal ideals, prove that I^n and J^n are co-maximal for each n.